Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

January 2016: Problem 2 Solution

Exercise. Let $[a, b]$ be a (bounded) interval of $\mathbb{R}$ and let m be Lebesgue measure. Let M be a positive real number and let $f_1, f_2, \ldots$ be a sequence of measurable functions on $[a, b]$ for which $\int_a^b |f_n| dm \leq M$ for every n. Assume that $f_n(x) \to f(x)$ as $n \to \infty$ for m-almost every x.

(a) State Fatou's lemma.

Solution.

Fatou's Lemma: If $\{f_n\}$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n$$

(b) Show that $\int_a^b |f| dm \leq M$.

Solution.

Since $f_1, f_2, \ldots$ measurable, so are $|f_1|, |f_2|, \ldots$

$\implies \{|f_n|\} \in L^+$

$$\int_a^b |f_n| dm \leq M \qquad \text{for all } n$$

$$\implies \quad \limsup \int_a^b |f_n| dm \leq M$$

$$\text{And} \quad \liminf |f_n| = \lim |f_n| = |f|$$

$$\begin{aligned} \implies \quad \int_a^b |f| dm &= \int_a^b \liminf |f_n| dm \\ &\leq \liminf \int_a^b |f_n| dm, \qquad \text{by Fatou's Lemma} \\ &\leq M \end{aligned}$$

(c) Suppose that $||f_n - f_k||_1 \to 0$. Prove for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset [a, b]$ is m-measurable and $m(A) \leq \delta$, then $\int_A |f_n| dm \leq \epsilon$ for all n.

Solution.

Let $\epsilon > 0$. By part (b), $f \in L^1$.

$\implies \exists \delta_0$ s.t. if $m(A) < \delta_0$ for measurable subset $A \subset [a, b]$

$$\int_A |f| dm \leq \frac{\epsilon}{2}$$

Since $||f_n - f||_1 \to 0$ as $n \to \infty$, $\exists N \in N$ s.t. $\forall n \geq N$,

$$\int |f_n(x) - f(x)| dm \leq \frac{\epsilon}{2}$$

Since $f_n \in L^1$, $\exists \delta_m > 0$ s.t. if $m(A) < \delta_m$ for measurable subset $A \subset [a, b]$

$$\int_A |f_n| dm \leq \epsilon$$

Let $\delta = \min\{d_0, \ldots, d_N\} \geq 0$. We have that for all measurable subsets $A \subset [a, b]$ with $m(A) < \delta$

$$\int_A |f_n| dm \leq \int_A |f_n - f| dm + \int_A |f| dm \leq \epsilon$$