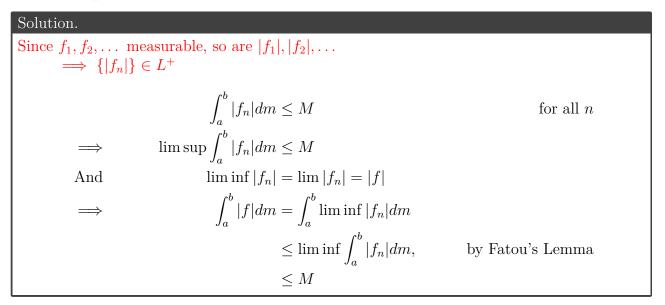
## Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam January 2016: Problem 2 Solution

**Exercise.** Let [a, b] be a (bounded) interval of  $\mathbb{R}$  and let m be Lebesgue measure. Let M be a positive real number and let  $f_1, f_2, \ldots$  be a sequence of measurable functions on [a, b] for which  $\int_a^b |f_n| dm \leq M$  for every n. Assume that  $f_n(x) \to f(x)$  as  $n \to \infty$  for m-almost every x.

(a) State Fatou's lemma.

Solution.
<b><u>Fatou's Lemma</u></b> : If $\{f_n\}$ is any sequence in $L^+$ , then
$\int (\liminf f_n) \le \liminf \int f_n$

(b) Show that  $\int_a^b |f| dm \le M$ .



(c) Suppose that  $||f_n - f_k||_1 \to 0$ . Prove for every  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $A \subset [a, b]$  is *m*-measurable and  $m(A) \leq \delta$ , then  $\int_A |f_n| dm \leq \epsilon$  for all *n*.

## Solution.

Let  $\epsilon > 0$ . By part (b),  $f \in L^1$ .  $\implies \exists \delta_0 \text{ s.t. if } m(A) < \delta_0 \text{ for measurable subset } A \subset [a, b]$ 

$$\int_A |f| dm \le \frac{\epsilon}{2}$$

Since  $||f_n - f||_1 \to 0$  as  $n \to \infty$ ,  $\exists N \in N$  s.t.  $\forall n \ge N$ ,

$$\int |f_n(x) - f(x)| dm \le \frac{\epsilon}{2}$$

Since  $f_n \in L^1$ ,  $\exists \delta_m > 0$  s.t. if  $m(A) < \delta_m$  for measurable subset  $A \subset [a, b]$ 

$$\int_A |f_n| dm \le \epsilon$$

Let  $\delta = \min\{d_0, \dots, d_N\} \ge 0$ . We have that for all measurable subsets  $A \subset [a, b]$  with  $m(A) < \delta$  $\int_A |f_n| dm \le \int_A |f_n - f| dm + \int_A |f| dm \le \epsilon$